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OPTIMAL CONTROL OF MHD-FLOW DECELERATION

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A problem of pulsed control for a three-dimensional magnetohydrodynamic (MHD) *model is considered. It is demonstrated that singularities of the solution of* MHD *equations do not develop with time because they are suppressed by a magnetic field. The existence of an optimal control is proved. An optimality system with the solution regular in time as a whole is constructed.*

Key words: *magnetohydrodynamic equations, pulsed control, optimality conditions.*

Introduction. A flow of a homogeneous viscous incompressible conducting fluid in a bounded, simply connected domain $\Omega \subset \mathbb{R}^3$ with a connected boundary $\Gamma = \partial\Omega$ is modeled by magnetohydrodynamic (MHD) equations in dimensionless variables:

$$\operatorname{div} \boldsymbol{u} = 0, \qquad \operatorname{div} \boldsymbol{B} = 0; \tag{1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + S\cdot\operatorname{rot}\boldsymbol{B}\times\boldsymbol{B} + \boldsymbol{q}, \qquad x\in\Omega, \quad t\in(0,T); \tag{2}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \operatorname{rot}\boldsymbol{E} = 0, \qquad \operatorname{rot}\boldsymbol{B} = \frac{1}{\nu_m}\,(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}). \tag{3}$$

Here $\boldsymbol{u}$, $\boldsymbol{B}$, and $\boldsymbol{E}$ are the vector fields of velocity, magnetic induction, and electric strength, respectively, p is the pressure, $\boldsymbol{q} = \boldsymbol{q}(x)$ is the density of external forces, $\nu = 1/\operatorname{Re}$, $\nu_m = 1/\operatorname{Re}_m$, $S = M^2/(\operatorname{Re}\operatorname{Re}_m)$, Re is the Reynolds numbers, Re_m is the magnetic Reynolds number, and M is the Hartmann number.

Equations (1)–(3) are supplemented by the conditions on the boundary Γ of the flow domain

$$\boldsymbol{u} = 0, \quad \boldsymbol{B}\cdot\boldsymbol{n} = 0, \quad \boldsymbol{n}\times\boldsymbol{E} = 0, \qquad (x,t)\in\Gamma\times(0,T) \tag{4}$$

($\boldsymbol{n}$ is the unit vector of the external normal to the boundary Γ) and by the initial conditions

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0(x), \quad \boldsymbol{B}|_{t=0} = 0, \qquad x\in\Omega. \tag{5}$$

A method of flow deceleration with the use of pulsed control by a magnetic field is proposed. The control functions are chosen to be the jumps $\boldsymbol{b}_i$ of the magnetic field at the times $0 < t_1 < t_2 < \ldots < t_m < T$. In this case, the MHD flow is described by Eqs. (1) and (3) and by the equations

$$\frac{\partial \boldsymbol{B}}{\partial t} + \operatorname{rot}\boldsymbol{E} = \sum_{i=1}^{m}\delta(t-t_i)\boldsymbol{b}_i, \qquad \operatorname{rot}\boldsymbol{B} = \frac{1}{\nu_m}\,(\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B})$$

with the initial-boundary conditions (4) and (5). Here $\delta(t-t_i)$ is the Dirac δ-function with the support at the point t_i.

The task is to minimize the functional

$$J = \frac{1}{4}\int_0^T\int_\Omega ((\operatorname{rot}\boldsymbol{u})^2 + (\operatorname{rot}\boldsymbol{B})^2)^2\,dx\,dt + \frac{\lambda}{2}\sum_{i=1}^{m}\int_\Omega(\operatorname{rot}\boldsymbol{b}_i)^2\,dx$$

($\lambda > 0$ is the regularization parameter).

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The classical boundary-value problems for the evolutionary model (1)–(3) were studied in [1, 2]. The problems of the optimal control of the evolutionary Navier–Stokes systems were first studied in [3–5]. The optimal control of unsteady MHD equations was considered in [6]. In studying the problems of optimal control of three-dimensional systems of Navier–Stokes-type equations, the main problem is to ensure regularity of the optimal state of the flow. For the statement examined, it is demonstrated that singularities of the solution (in the Leray sense) do not develop with time because they are suppressed by a magnetic field. Solvability of the control problem is proved. An optimality system is constructed, and its regularity in time as a whole is justified. The method of deriving the optimality conditions is close to the method proposed in [7].

1. Formalization and Solvability of the Control Problem. To simplify transformations, we use the re-normalization

$$\boldsymbol{B} = \sqrt{S}\,\boldsymbol{B}, \qquad \boldsymbol{E} = \sqrt{S}\,\boldsymbol{E}.$$

Then, system (1)–(3) acquires the form

$$\operatorname{div}\boldsymbol{u} = 0, \qquad \operatorname{div}\boldsymbol{B} = 0;$$

$$\boldsymbol{u}' - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + \operatorname{rot}\boldsymbol{B}\times\boldsymbol{B} + \boldsymbol{q}, \qquad x \in \Omega, \quad t \in (0,T);$$

$$\boldsymbol{B}' + \operatorname{rot}\boldsymbol{E} = 0, \qquad \boldsymbol{E} = \nu_m \operatorname{rot}\boldsymbol{B} - \boldsymbol{u}\times\boldsymbol{B},$$

where $\boldsymbol{u}' = \partial\boldsymbol{u}/\partial t$ and $\boldsymbol{B}' = \partial\boldsymbol{B}/\partial t$.

We consider the spaces of vector functions and operators necessary to analyze the control problem [2]. Let Ω be a simply connected domain in the space $\mathbb{R}^3$ with a connected boundary $\Gamma \in C^2$. We introduce the spaces

$$U_1 = \{\boldsymbol{v} \in C^\infty(\bar\Omega)\colon \quad \operatorname{div}\boldsymbol{v} = 0,\ x \in \Omega, \quad \boldsymbol{v} = 0,\ x \in \Gamma\},$$

$$U_2 = \{\boldsymbol{v} \in C^\infty(\bar\Omega)\colon \quad \operatorname{div}\boldsymbol{v} = 0,\ x \in \Omega, \quad \boldsymbol{n}\cdot\boldsymbol{v} = 0,\ x \in \Gamma\},$$

V_1 is the closure of U_1 with the norm $W_2^1(\Omega)$, V_2 is the closure of U_2 with the norm $W_2^1(\Omega)$, H_1 is the closure of U_1 with the norm $L^2(\Omega)$, and H_2 is the closure of U_2 with the norm $L^2(\Omega)$.

We assume that

$$(\boldsymbol{u},\boldsymbol{v})_0 = \int\limits_\Omega (\boldsymbol{u}\cdot\boldsymbol{v})\,dx$$

is a scalar product in the spaces H_1 and H_2,

$$((\boldsymbol{u},\boldsymbol{v})) = (\operatorname{rot}\boldsymbol{u},\operatorname{rot}\boldsymbol{v})_0 = \int\limits_\Omega (\operatorname{rot}\boldsymbol{u}\cdot\operatorname{rot}\boldsymbol{v})\,dx \qquad \forall\,\boldsymbol{u},\boldsymbol{v} \in V_1, V_2$$

is a scalar product in the spaces V_1 and V_2, and the norm determined by this product is equivalent to the space norm $W_2^1(\Omega)$. Let X be a Banach space. Then, $L^p(0,T;X)$ ($C([0,T];X)$ designates a space L^p (class C) of functions determined on the interval $[0,T]$ with their values in the space X. We determine the spaces

$$V = V_1\times V_2, \qquad H = H_1\times H_2, \qquad V \subset H = H' \subset V'.$$

This embedding of spaces is dense and continuous. The norms in the spaces V and H and in the adjoint space V' are denoted by $\|\cdot\|$, $|\cdot|$, and $\|\cdot\|_*$, respectively; $(\,\cdot\,,\,\cdot\,)$ is the duality relation between V' and V and a scalar product in H:

$$(y,z) = (\boldsymbol{u},\boldsymbol{v})_0 + (\boldsymbol{B},\boldsymbol{w})_0, \quad (y,z)_V = ((\boldsymbol{u},\boldsymbol{v})) + ((\boldsymbol{B},\boldsymbol{w})) \quad \forall\, y = \{\boldsymbol{u},\boldsymbol{B}\},\ z = \{\boldsymbol{v},\boldsymbol{w}\}.$$

We introduce the mappings $A_1\colon V_1 \to V_1'$, $A_2\colon V_2 \to V_2'$, $A\colon V \to V'$, and $B\colon V\times V \to V'$, using the relations

$$(Ay,z) = \nu((\boldsymbol{u},\boldsymbol{v})) + \nu_m((\boldsymbol{B},\boldsymbol{w})) = \nu(A_1\boldsymbol{u},\boldsymbol{v}) + \nu_m(A_2\boldsymbol{B},\boldsymbol{w}),$$

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \int\limits_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}\,dx,$$

$$(B(y_1,y_2),z) = b(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{v}) - b(\boldsymbol{B}_1,\boldsymbol{B}_2,\boldsymbol{v}) + b(\boldsymbol{B}_1,\boldsymbol{B}_2,\boldsymbol{w}) - b(\boldsymbol{B}_1,\boldsymbol{u}_2,\boldsymbol{w}),$$

which are satisfied for all $y = \{\boldsymbol{u}, \boldsymbol{B}\}$, $y_1 = \{\boldsymbol{u}_1, \boldsymbol{B}_1\}$, $y_2 = \{\boldsymbol{u}_2, \boldsymbol{B}_2\}$, and $z = \{\boldsymbol{v}, \boldsymbol{w}\}$ from the space V. The operator A satisfies the conditions

$$(Ay, y) \geq \alpha\|y\|^2, \quad \alpha = \min(\nu, \nu_m), \qquad (Ay, z) = (Az, y) \quad \forall\, y, z \in V.$$

We also note that $(B(y, z), z) = 0$, and the mapping $B[y] = B(y, y)$ is compact. Let $D(A) = \{y \in V\colon\ Ay \in H\}$. The following estimates are valid for the operator $B(y, z)$ [2]:

$$(B(y_1, y_2), y_3) \leq C|y_1|^{1/4}\|y_1\|^{3/4}\|y_2\||y_3|^{1/4}\|y_3\|^{3/4}, \quad y_1 \in V,\ y_2 \in V,\ y_3 \in V; \tag{6}$$

$$(B(y_1, y_2), y_3) \leq C\|y_1\|\|y_2\|^{1/2}|Ay_2|^{1/2}|y_3|^{1/2}, \quad y_1 \in V,\ y_2 \in D(A),\ y_3 \in H. \tag{7}$$

Here the constant $C > 0$ depends only on Ω, Re, and Re_m.

Determining the functional $Q \in V'$, $(Q, z) = (\boldsymbol{q}, \boldsymbol{v})_0$, where $z = \{\boldsymbol{v}, \boldsymbol{w}\}$, we can reduce the initial-boundary problem (1)–(5) to the Cauchy problem for the equation with operator coefficients [2]

$$y' + Ay + B[y] = Q, \qquad y(0) = y_0,$$

where $y_0 = \{\boldsymbol{u}_0, 0\}$.

Let us consider the division of the time interval $(0, T)$ into segments $t_0 = 0 < t_1 < t_2 < \ldots < t_m < T = t_{m+1}$. We find the admissible vector of pulsed control

$$f = (\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_m) \in V_2 \times V_2 \times \ldots \times V_2 = U$$

and the state of the system

$$y = \{\boldsymbol{u}, \boldsymbol{B}\} \in L^\infty(0, T; V),$$

$$y\Big|_{[t_{i-1}, t_i]} \in C([t_{i-1}, t_i]; H), \quad y'\Big|_{[t_{i-1}, t_i]} \in L^2(t_{i-1}, t_i; H), \quad i = 1, 2, \ldots, m+1,$$

such that

$$y' + Ay + B[y] = Q + \sum_{i=1}^{m} \delta(t - t_i)\zeta_i, \qquad y(0) = y_0. \tag{8}$$

Here $\zeta_i = \{0, \boldsymbol{b}_i\}$ and $\delta(t - t_i)$ is the Dirac δ-function with the support at the point t_i.

The essence of the optimal control problem is finding an admissible pair $\{f, y\}$ minimizing the functional

$$J(f, y) = \frac{1}{4}\int_0^T \|y(t)\|^4\, dt + \frac{\lambda}{2}\, \|f\|_U^2. \tag{9}$$

Here $\|f\|_U^2 = \sum_{i=1}^{m} \|\boldsymbol{b}_i\|_{V_2}^2$.

Remark 1. Problem (8) actually means that

$$y' + Ay + B[y] = Q, \quad t \in (t_{i-1}, t_i), \qquad y(t_i + 0) - y(t_i - 0) = \{0, \boldsymbol{b}_i\}, \quad i = 1, \ldots, m.$$

The set of admissible pairs is not empty if, for instance, the solution $\boldsymbol{u}$ of the system of Navier–Stokes equations with the initial condition (5) for a zero magnetic field belongs to the class $L^4(0, T; V_1)$. Then, the pair $y = \{\boldsymbol{u}, 0\}$, $f = 0$ is admissible, which is valid, in particular, if $\boldsymbol{u}_0 \in V_1$ is a steady-state solution of the Navier–Stokes equations, which can be unstable.

Let us find the *a priori* estimates for the solution of the controlled system (8). The energy inequality yields the estimate

$$|y(t)|^2 + \int_0^{t_1} \|y(\tau)\|^2 d\tau \leq Ct_1\|Q\|_*^2 + \|\mathbf{u}_0\|_{L^2(\Omega)}^2 \tag{10}$$

on the interval $(0, t_1)$, the estimate

$$|y(t)|^2 + \int_{t_1}^{t_2} \|y(\tau)\|^2\, d\tau \leq C(t_2 - t_1)\|Q\|_*^2 + \|\boldsymbol{u}|_{t=t_1}\|_{L^2(\Omega)}^2 + \|\boldsymbol{B}\Big|_{t=t_1-0} + \boldsymbol{b}_1\|_{L^2(\Omega)}^2 \tag{11}$$

on the interval (t_1, t_2), etc. As in other *a priori* estimates, $C, C_1, \dots$ here denote positive constants depending only on Ω and dimensionless parameters Re, Re_m, and S. It follows from Eqs. (10) and (11) that

$$|y(t)|^2 + \int\limits_0^T \|y(\tau)\|^2 \, d\tau \le C(\|Q\|_*^2 + \|\boldsymbol{u}_0\|^2_{L^2(\Omega)} + |f|^2) \tag{12}$$

on the interval $(0, T)$. Here $|f|^2 = \sum\limits_{i=1}^{m} \|\boldsymbol{b}_i\|^2_{L^2(\Omega)}$.

If $\boldsymbol{q} \in L^2(\Omega)$, the following inequality is valid on each of the intervals (t_{i-1}, t_i) [2]:

$$\frac{d}{dt}\|y\|^2 + |Ay|^2 \le C_1(\|y\|^6 + |Q|). \tag{13}$$

Estimate (13) implies the boundedness of the state y in the space $L^\infty(0, T; V)$ and of the state Ay in the space $L^2(0, T; V)$, if the state y is bounded in the space $L^4(0, T; V)$ and the pulsed control is bounded in V_2.

Theorem 1. *Let the vector field* $\boldsymbol{u}_0 \in V_1$ *be a steady-state solution of the system of the Navier–Stokes equations*

$$-\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + \boldsymbol{q}, \qquad \operatorname{div}\boldsymbol{u} = 0, \quad x \in \Omega, \qquad \boldsymbol{u}\Big|_\Gamma = 0,$$

where $\boldsymbol{q} \in L^2(\Omega)$*. Then, there exists a solution of problem* (9).

Proof. As was noted in Remark 1, the set of admissible pairs is not empty is the theorem conditions are satisfied. The following inequality is valid for the sequence of admissible pairs $\{f_k, y_k\}$ minimizing the functional J:

$$\frac{1}{4}\int\limits_0^T \|y_k(t)\|^4 \, dt + \frac{\lambda}{2}\|f_k\|_U^2 \le C$$

(the constant C is independent of k). Passing, if necessary, to a subsequence, we conclude that $f_k \to f_*$ weakly in U and $y_k \to y_*$ weakly in $L^4(0, T; V)$.

Based on estimates (12) and (13) we obtain the following properties of the sequence y_k: y_k is bounded in $L^\infty(0, T; V)$ and Ay_k is bounded in $L^2(0, T; H)$. Then, it follows from Eq. (8) that y_k' is bounded in $L^2(t_{i-1}, t_i; H)$. The boundedness of the sequences y_k, Ay_k, and y_k' is a sufficient condition for the limiting transition in Eq. (8) and for justifying that $\{f_*, y_*\}$ is an admissible pair. The optimality of this pair is a consequence of weak lower semi-continuity of the functional J.

2. Optimality System. Deriving necessary first-order conditions of the extremum in control problems for nonlinear systems usually requires additional conditions of regularity of the set of admissible pairs or the optimal state (see [5]). For systems of equations of the Navier–Stokes type, the solution regularity is justified only locally in time or under the condition that the initial data are small [8]. In the problem considered here, the necessary regularity of the optimal flow is a consequence of the structure of the functional J.

Let a pair $\{f_*, y_*\}$ be a solution of the control problem (9). Then, the optimality of this pair yields the inequality

$$\frac{1}{4}\int\limits_0^T \|y_*(t)\|^4 \, dt + \frac{\lambda}{2}\|f_*\|_U^2 \le \frac{1}{4}T\|\boldsymbol{u}_0\|_{V_1},$$

which means, by virtue of estimate (13), that

$$\|y_*(t)\| \le C, \quad t \in (0, T), \qquad \int\limits_0^T \|Ay_*(t)\|^2 \, dt \le C. \tag{14}$$

We denote $F(y, z) = B(y, z) + B(z, y)$ and determine the bilinear mapping $F'(y, z)$:

$$(F'(y, z), z_1) = (F(y, z_1), z) \qquad \forall\, y, z, z_1 \in V.$$

To construct the optimality system, we fix an arbitrary element $\tilde{f} = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_m\}$ from the control space U and consider the following problem with a small parameter $\varepsilon > 0$.

We have to find functions

$$z_\varepsilon \in L^2(0,T;V), \quad z'_\varepsilon \in L^2(t_{i-1},t_i;H), \quad i=1,2,\dots,m+1,$$

such that

$$z'_\varepsilon + Az_\varepsilon + F(y_*,z_\varepsilon) + \varepsilon B[z_\varepsilon] = \sum_{i=1}^{m} \delta(t-t_i)\{0,\boldsymbol{w}_i\}, \qquad z_\varepsilon(0)=0. \tag{15}$$

Let us find some *a priori* estimates of the solution of problem (15). On the interval $(0,t_1)$, we multiply the equation in (15) scalarly by z_ε and integrate it with respect to t with allowance for the initial condition. Based on inequality (6) for the nonlinear operator B and taking into account the boundedness of the optimal state y_*, we obtain

$$|z_\varepsilon(t)|^2 + \int_0^t \|z_\varepsilon(\tau)\|^2\,d\tau \le C\int_0^t |z_\varepsilon(\tau)|^{1/2}\|z_\varepsilon(\tau)\|^{3/2}\,d\tau.$$

Applying Young's inequality to the integral in the right side and using Gronwall's lemma, we find $z_\varepsilon = 0$ on the interval $(0,t_1)$. Similarly, on the interval (t_1,t_2), with allowance for the condition $z_\varepsilon(t_1)=\{0,\boldsymbol{w}_1\}$, we derive the inequality

$$|z_\varepsilon(t)|^2 + \int_{t_1}^{t_2} \|z_\varepsilon(\tau)\|^2\,d\tau \le C\|\boldsymbol{w}_1\|^2_{H_2}.$$

Obtaining the same estimates of the solution z_ε on the remaining intervals, we have the following inequality on the entire interval $(0,T)$:

$$|z_\varepsilon(t)|^2 + \int_0^T \|z_\varepsilon(\tau)\|^2\,d\tau \le C|\tilde{f}|^2_U. \tag{16}$$

Let us find the estimates for z_ε in the space $L^\infty(0,T;V)$ and Az_ε in the space $L^2(0,T;H)$. On the interval (t_1,t_2), we multiply the equation in (15) scalarly by Az_ε, assuming (without loss of generality) that $\nu=1$ and $\nu_m=1$. Then, we have

$$\frac{1}{2}\frac{d}{dt}\|z_\varepsilon\|^2 + |Az_\varepsilon|^2 = -(F(y_*,z_\varepsilon),Az_\varepsilon) - \varepsilon(B[z_\varepsilon],Az_\varepsilon). \tag{17}$$

To obtain an estimate of the right side of Eq. (17), we apply inequality (7), again with allowance for the boundedness of y_* in the space $L^\infty(0,T;V)$:

$$|(F(y_*,z_\varepsilon),Az_\varepsilon)| \le C(\|z_\varepsilon\|^{1/2}|Az_\varepsilon|^{3/2} + \|z_\varepsilon\||Az_\varepsilon||Ay_*|^{1/2}),$$

$$|(B[z_\varepsilon],Az_\varepsilon)| \le C\|z_\varepsilon\|^{3/2}|Az_\varepsilon|^{3/2}.$$

By virtue of Young's inequality, Eq. (17) yields the estimate

$$\frac{d}{dt}\|z_\varepsilon\|^2 + |Az_\varepsilon|^2 \le C_1(1+|Ay_*|)\|z_\varepsilon\|^2 + C_2\varepsilon\|z_\varepsilon\|^6. \tag{18}$$

We define the functions

$$\alpha(t) = C_1(1+|Ay_*(t)|), \qquad \beta(t) = \exp\Big(2\int_{t_1}^t \alpha(\tau)\,d\tau\Big), \qquad \gamma(t) = \int_{t_1}^t \beta(\tau)\,d\tau,$$

where C_1 and C_2 are constants in Eq. (18), which do not depend on ε. Integrating the differential inequality (18), we obtain the estimate

$$\|z_\varepsilon(t)\|^4 \le \|\boldsymbol{w}_1\|^4\beta(t)/(1-2C_2\varepsilon\|\boldsymbol{w}_1\|^4\gamma(t)),$$

which implies that

$$\|z_\varepsilon(t)\|^4 \le 2\|\boldsymbol{w}_1\|^4\beta(t_2),$$

if $2C_2\varepsilon\|\boldsymbol{w}_1\|^4\gamma(t_2) \le 1/2$. Thus, for a sufficiently small parameter ε, it follows from Eq. (18) that Az_ε is bounded in the space $L^2(t_1, t_2; H)$. Similar estimates of the function z_ε on other intervals with small values of ε yield the inequality

$$\|z_\varepsilon(t)\|^2 + \int\limits_0^T |Az_\varepsilon(\tau)|^2\, d\tau \le K, \tag{19}$$

where the constant K depends only on $\tilde{f}$ and y_*, and the initial data Ω, $\boldsymbol{q}$, T, Re, Re_m, and S.

Estimates (16)–(19) are sufficient to justify the solvability of problem (15). Note also that the pair $\{f_* + \varepsilon\tilde{f}, y_* + \varepsilon z_\varepsilon\}$ is admissible for the control problem; hence, we obtain

$$\varepsilon^{-1}(J(f_* + \varepsilon\tilde{f}, y_* + \varepsilon z_\varepsilon) - J(f_*, y_*)) \ge 0. \tag{20}$$

Inequalities (16)–(19) allow us to perform a limiting transition in Eq. (15) and inequality (20), and similar estimates are valid for the limiting function z. Thus, we obtain the following statement.

Lemma 1. *Let $\{f_*, y_*\}$ be the optimal pair, and $f_* = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_m\}$. Then, for each element $\tilde{f} = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\} \in U$, there exists a function*

$$z \in L^2(0, T; V), \quad z' \in L^2(t_{i-1}, t_i; H), \quad i = 1, 2, \ldots, m+1,$$

such that

$$z' + Az + F(y_*, z) = \sum_{i=1}^m \delta(t - t_i)\{0, \boldsymbol{w}_i\}, \qquad z(0) = 0; \tag{21}$$

$$\int\limits_0^T \|y_*(t)\|^2 (Ay_*(t), z(t))\, dt + \lambda \sum_{i=1}^m (A_2\boldsymbol{b}_i, \boldsymbol{w}_i) = 0. \tag{22}$$

Let us formulate a statement about the existence of an adjoint state.

Lemma 2. *Let $y_* \in L^\infty(0, T; V)$ and $Ay_* \in L^2(0, T; H)$. Then, there exists a unique solution of the adjoint system*

$$\begin{gathered} g \in L^2(0, T; V), \qquad z' \in L^2(0, T; V'), \\ -g' + Ag + F(y_*, g) = \|y_*\|^2 Ay_*, \qquad g(T) = 0. \end{gathered} \tag{23}$$

The proof of Lemma 2 is based on estimate (14) and is similar to that described in [7].

Theorem 2. *For each optimal pair $\{f_*, y_*\}$, there exists a unique solution of the adjoint system* (23)*, the pulsed controls being determined by the equalities*

$$\boldsymbol{b}_i = -\lambda^{-1} A_2^{-1}(Pg(t_i)), \qquad i = 1, \ldots, m.$$

Here P is the projecting operator; $Pg = \boldsymbol{r}$ if $g = \{\boldsymbol{h}, \boldsymbol{r}\}$.

The proof of Theorem 2 follows from Lemma 1 and Lemma 2, if Eq. (23) is scalarly multiplied by the function z satisfying condition (21) and Eq. (22) is taken into account.

Remark 2. A steady flow $\boldsymbol{u}_0$ for which Theorem 1 was formulated can turn out to be unstable; even in this case, however, the pulsed controls determined by Theorem 2 allow smooth solutions of the system of MHD equations and the optimality system to be obtained.

Thus, the optimal pair with the corresponding adjoint state is a solution of the optimality system, which is a nonlocal initial-boundary problem:

$$y_*' + Ay_* + B[y_*] = Q + \sum_{i=1}^m \delta(t - t_i)\{0, \boldsymbol{b}_i\}, \qquad y_*(0) = \{\boldsymbol{u}_0, 0\}; \tag{24}$$

$$-g' + Ag + F(y_*, g) = \|y_*\|^2 Ay_*, \qquad g(T) = 0; \tag{25}$$

$$\lambda A_2\boldsymbol{b}_i + Pg(t_i) = 0, \qquad i = 1, \ldots, m. \tag{26}$$

Let us give an interpretation of the optimality system (24)–(26). We consider the adjoint system (25). Let $g = \{\boldsymbol{h}, \boldsymbol{r}\}$ be an adjoint state corresponding to the optimal pair $y_* = \{\boldsymbol{u}_*, \boldsymbol{B}_*\}$. This system is a weak formulation of the following problem:

$$-\frac{\partial \boldsymbol{h}}{\partial t} - \nu \Delta \boldsymbol{h} + \boldsymbol{h} \times \operatorname{rot} \boldsymbol{u}_* + \operatorname{rot} \boldsymbol{r} \times \boldsymbol{B}_* + \operatorname{rot} (\boldsymbol{u}_* \times \boldsymbol{h}) = -\nabla \eta - \|y_*\|^2 \Delta \boldsymbol{u}_*,$$

$$-\frac{\partial \boldsymbol{r}}{\partial t} + \operatorname{rot} D + \operatorname{rot} \boldsymbol{B}_* \times \boldsymbol{h} + \boldsymbol{u}_* \times \operatorname{rot} \boldsymbol{r} = -\nabla \mu - \|y_*\|^2 \Delta \boldsymbol{B}_*,$$

$$D = \nu_m \operatorname{rot} \boldsymbol{r} + \boldsymbol{h} \times \boldsymbol{B}_*, \qquad \operatorname{div} \boldsymbol{h} = 0, \qquad \operatorname{div} \boldsymbol{r} = 0,$$

$$\boldsymbol{h} = 0, \quad \boldsymbol{r} \cdot \boldsymbol{n} = 0, \quad \boldsymbol{n} \times D = 0, \qquad (x,t) \in \Gamma \times (0,T).$$

Here η and μ are certain scalar functions.

The following conditions are satisfied at the time $t = T$:

$$\boldsymbol{h}\Big|_{t=T} = 0, \qquad \boldsymbol{r}\Big|_{t=T} = 0.$$

The pulsed controls $\boldsymbol{b}_i$ are solutions of the system of the Stokes equations

$$-\lambda \Delta \boldsymbol{b}_i + \nabla \delta_i = -\boldsymbol{r}\Big|_{t=t_i}, \quad \operatorname{div} \boldsymbol{b}_i = 0 \quad \text{in } \Omega, \qquad \boldsymbol{b}_i \cdot \boldsymbol{n} = 0, \quad \delta_i = 0 \quad \text{on } \Gamma.$$

The resultant nonlocal boundary-value problem whose solution is regular in time as a whole can serve as the basis for the subsequent qualitative and numerical analysis of the pulsed control problem considered.

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